

3.3 Phase portraits

$$\vec{x}'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = A \vec{x}, \quad A \text{ is a constant } 2 \times 2 \text{ matrix.}$$

Phase portrait: several representative curves (including the equilibria) on the x_1 - x_2 plane, that help to understand the asymptotic behavior of the solutions (as $t \rightarrow \pm\infty$)

Rk: the stability of the equilibria is easy to read on the phase portrait.

In section 3.3: the eigenvalues of A (λ_1, λ_2) are real.
(complex case \Rightarrow section 3.4)

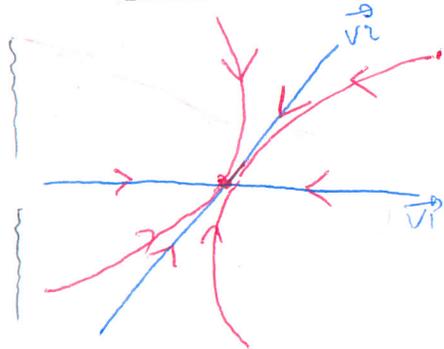
① IF $\lambda_1, \lambda_2 \neq 0$: A is invertible

$\rightarrow 0$ is the only equilibrium point.

1-1) IF $\lambda_1 \neq \lambda_2$

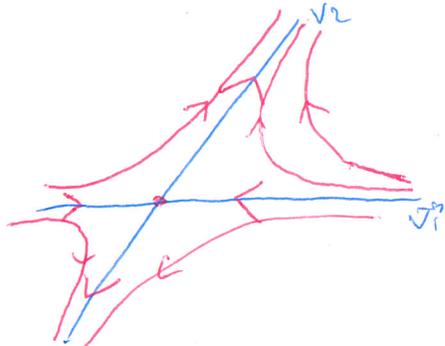
General form of the solution: $\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$
eigenvector for λ_1 eigenvector for λ_2

$$\lambda_1 < \lambda_2 < 0$$



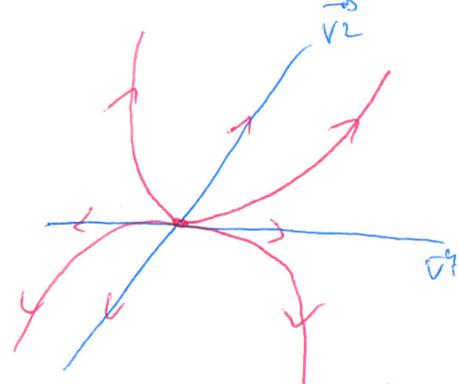
Attractive improper node
(stable)

$$\lambda_1 < 0 < \lambda_2$$



Saddle
(unstable)

$$0 < \lambda_1 < \lambda_2$$

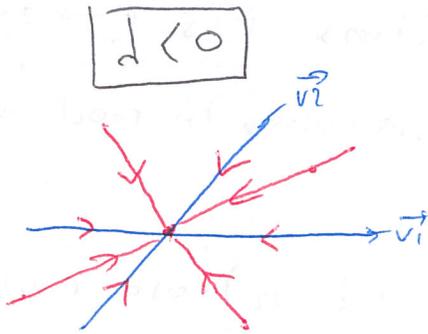


Repulsive improper node
(unstable)

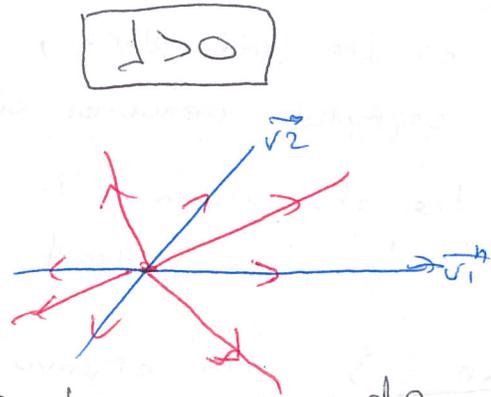
1-2) IF $\lambda_1 = \lambda_2$ ($\lambda = \lambda_1 = \lambda_2$)

1-2-1) IF A is diagonalizable

General form of the solution: $\vec{x}(t) = c_1 e^{\lambda t} \vec{v}_1 + c_2 e^{\lambda t} \vec{v}_2$
 (\vec{v}_1 and \vec{v}_2 form a basis of eigenvectors of A)



Attractive proper node
(stable)



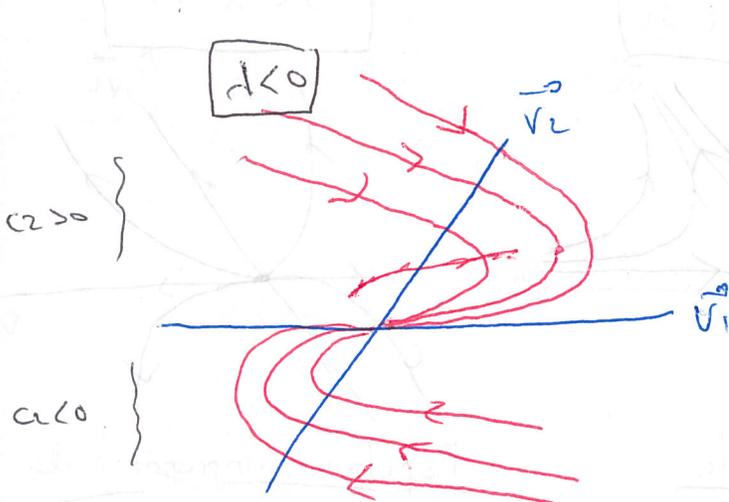
Repulsive proper node
(unstable)

1-2-2) IF A is not diagonalizable

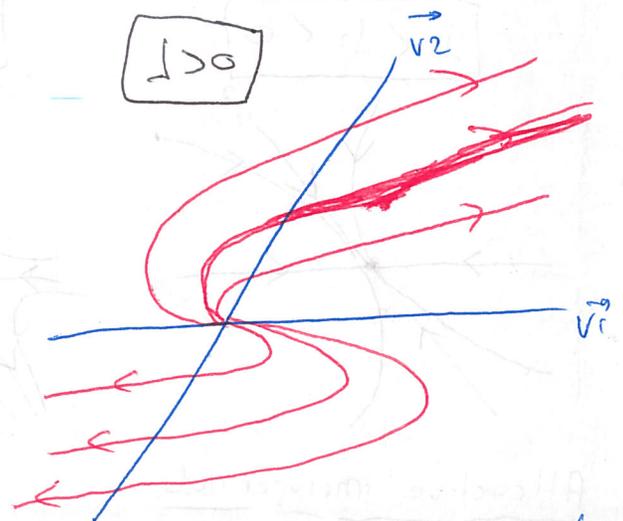
General form of the solution:

$$\vec{x}(t) = c_1 e^{\lambda t} \vec{v}_1 + c_2 e^{\lambda t} (\vec{v}_2 + t \vec{v}_1)$$

where: \vec{v}_1 is an eigenvector of A ,
 \vec{v}_2 is a solution of $A\vec{v}_2 = \lambda\vec{v}_2 + \vec{v}_1$



Attractive degenerate node (stable)



Repulsive degenerate node (unstable)

$$\vec{x}(t) = (c_1 + t c_2) e^{\lambda t} \vec{v}_1 + c_2 e^{\lambda t} \vec{v}_2$$

② IF at least one of the eigenvalues is 0.

IF $\lambda_1 = 0$:

2-0) IF A is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Every point is an equilibrium (constant solution)

Now: if $A \neq 0$: we have a line of equilibria

↓
this is $\text{Null } A \leftarrow \text{Eigenspace of } 0.$

2-1) IF $\lambda_2 \neq 0$

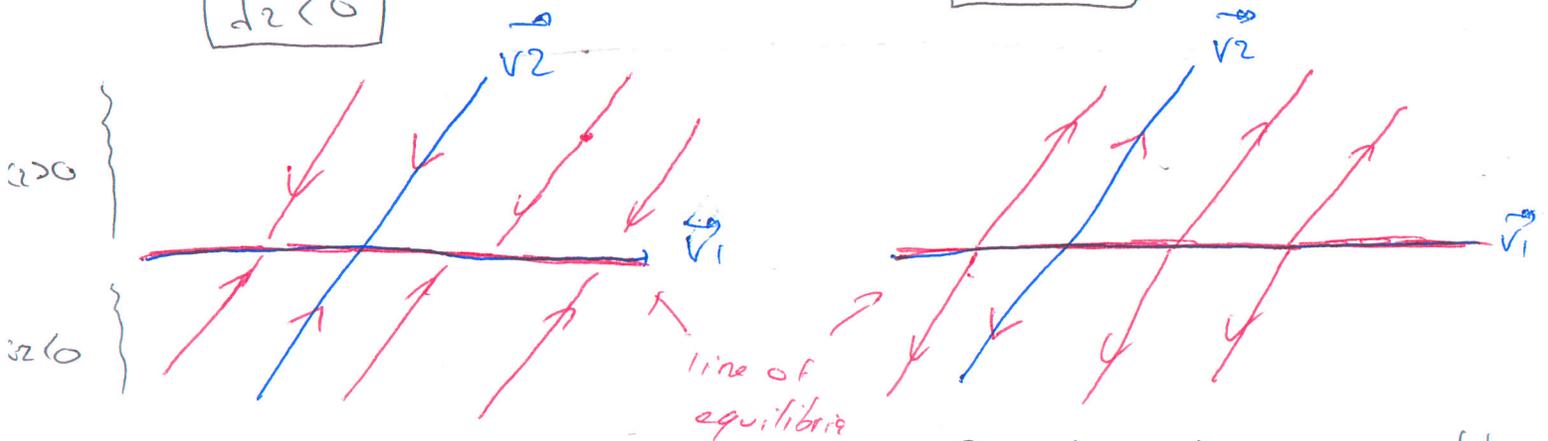
General form of the solutions:

$$\vec{x}(t) = c_1 \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$$

(\vec{v}_1 : eigenvector for $\lambda_1 = 0$
 \vec{v}_2 : eigenvector for $\lambda_2 \neq 0$)

$\lambda_2 < 0$

$\lambda_2 > 0$



Attractive line of equilibria
(stable)

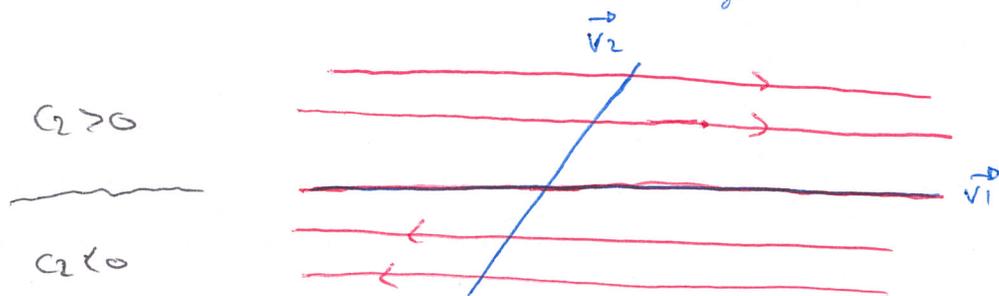
Repulsive line of equilibria
(unstable)

2-2) If $\lambda_1 = \lambda_2 = 0$ (but $A \neq 0$)

General form of the solution: $\vec{x}(t) = c_1 \vec{v}_1 + c_2 (\vec{v}_2 + t \vec{v}_1)$

eigenvector for λ_1

solution of $A\vec{v}_2 = \lambda_2 \vec{v}_2 + \vec{v}_1$



Laminated flow (unstable)

③ If the eigenvalues are complex: $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$, $\beta \neq 0$

General form of the solution: